University of Zurich

Department of Economics

Working Paper Series

ISSN 1664-7041 (print)
ISSN 1664-705X (online)

Working Paper No. 184

An Envelope Approach to Tournament Design

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Revised version, October 2015

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Abstract. Optimal rank-order tournaments have traditionally been studied using a first-order approach. The present analysis relies instead on the construction of an "upper envelope" over all incentive compatibility conditions. It turns out that the first-order approach is *not* innocuous. For example, in contrast to the traditional understanding, tournaments may be dominated by piece rates even if workers are risk-neutral. The paper also offers a strikingly simple characterization of the optimal tournament for quadratic costs and CARA utility, as well as an extension to large tournaments.

Keywords. Rank-order tournaments · First-order approach · Envelope theorem

JEL-Codes C62, D86, L23

*) The paper benefitted from the insightful comments of an anonymous referee. For useful discussions and comments, I would like to thank Tilman Börgers, Luis Corchón, Péter Esö, Ángel Hernando, René Kirkegaard, Kai Konrad, Dan Kovenock, Daniel Krähmer, Georg Nöldeke, Trond Olsen, David Peréz-Castrillo, Marco Serena (discussant), Karl Schmedders, and Felix Vardy. The present paper has been presented at the Zurich-Basel Workshop in Micro Theory, the Oligo Workshop in Madrid, the SAET Conference in Cambridge, the ESWC in Montréal, the EEA meetings in Mannheim, and the JEI in Alicante.

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1 Introduction

The economic analysis of rank-order tournaments presents itself today as a tremendously successful research area that has experienced a steady increase in interest since its very beginnings.[1] On the theoretical front, it has often been crucial to characterize the optimal tournament (Lazear and Rosen, 1981; Nalebuff and Stiglitz, 1983; Akerlof and Holden, 2012). This task has most commonly been accomplished using the so-called first-order approach, i.e., by replacing a continuum of incentive compatibility conditions in the firm's design problem with a single marginal condition. However, the first-order approach is not generally valid, and as a consequence, the properties of the optimal tournament have sometimes been discussed under somewhat restrictive or even indistinct conditions.[2]

In this paper, an alternative route to the analysis of optimal rank-order tournaments is taken. The approach entails the construction of an "upper envelope" over all incentive compatibility conditions, which is then added as an inequality constraint to the relaxed problem. Thereby, the optimal tournament may be characterized as the solution of an optimization problem with a finite number of constraints. Of course, the thereby reformulated problem remains difficult. However, in contrast to the original problem, techniques from Milgrom and Segal (2002) may be applied to derive key properties of the optimal tournament even if the first-order approach is invalid or difficult to justify.

[1] See, e.g., the evidence provided by Connelly et al. (2014). For an introduction to the economics of tournaments, see Mookherjee (1990) or Prendergast (1999, Sec. 2.3).

[2] Useful discussions of the scope and limitations of the first-order approach in tournament theory can be found in McLaughlin (1988) and Gürtler (2011).

The main result of this paper is that the first-order approach to tournament design is *not* innocuous. Specifically, it is found that traditional conclusions regarding the efficiency of rank-order tournaments are not universally valid and sometimes too optimistic. In fact, tournaments may be substantially less efficient than suggested by the existing literature.[3] Further, with additional structure imposed on the cost and utility functions, the optimal tournament may be characterized in explicit terms even if the first-order approach is invalid. The paper also considers an extension to tournaments with many contestants and a single winner, which may be seen as an equilibrium analysis complementing prior work.

The observation that the first-order approach is not generally valid in a moral hazard setting is due to Mirrlees (1975). Subsequent research on the first-order approach may be roughly divided into two strands. A first strand of literature is concerned with formulating sufficient conditions for the first-order approach (Rogerson, 1985; Jewitt, 1988; Sinclair-Desgagné, 1994; Conlon, 2009; Ke, 2013; Kirkegaard 2014a). A second strand of literature has aimed at eliminating restrictive assumptions from the standard model of moral hazard (Grossman and Hart, 1983; Mirrlees, 1986; Araujo and Moreira, 2001; Ke, 2012; Kadan and Swinkels, 2013; Kirkegaard 2014b; Renner and Schmedders, 2015). The present paper differentiates itself from these contributions already by its focus on rank-order tournaments. However, also the approach is different. For example, the present paper does not employ a Lagrangian function. Some implications of this point will be

[3] The present paper deals exclusively with the incentive side of rank-order tournaments. The selection efficiency of tournaments has been studied by Clark and Riis (2001), Hvide and Kristiansen (2003), and Münster (2007), for instance.

discussed in the conclusion.

The remainder of the paper is structured as follows. Section 2 introduces the set-up, and discusses existence. The envelope approach is developed in Section 3. A characterization of the optimal tournament is presented in Section 4, and discussed in Section 5. An extension with more than two contestants is offered in Section 6. Section 7 concludes. All proofs have been relegated to an Appendix.

2 Set-up and existence

Consider a market environment in which risk-neutral firms hire workers to produce output of per-unit value $V > 0$. Given a wage W and an effort level $\mu \geq 0$, a worker's utility is defined as $U(W) - C(\mu)$, where U is twice differentiable with $U' > 0$, $U'' \leq 0$, and C is four times differentiable with $C' \geq 0$, $C'' > 0$, $C'(0) = 0$, and $C'(\mu) \to \infty$ as $\mu \to \infty$.[4] Worker j's output ($j = 1, 2$) is the sum of his effort μ_j and a random component ε_j, i.e., $q_j = \mu_j + \varepsilon_j$. It will be assumed that the distribution function G of the differential error term $\xi \equiv \varepsilon_2 - \varepsilon_1$ is symmetric with respect to the origin and allows a twice differentiable density $g = G' > 0$ such that g' and g'' are bounded. Given a pair of prizes (W_1, W_2) with $W_1 \geq W_2$, worker j's expected

[4] The additively separable form of the utility function ensures tractability (cf. Green and Stokey, 1983; Krishna and Morgan, 1998; Akerlof and Holden, 2012). As discussed in McLaughlin (1988), alternative specifications of the worker's utility function tend to produce similar conclusions under the first-order approach. It is conjectured that the same is true for the additional settings considered in the present paper.

utility from playing μ_j against μ_k, with $k \neq j$, is then given as

$$\begin{aligned} & U(W_1)\mathrm{prob}[q_j > q_k] + U(W_2)(1 - \mathrm{prob}[q_j > q_k]) - C(\mu_j) \\ & = U(W_2) + (U(W_1) - U(W_2))G(\mu_j - \mu_k) - C(\mu_j). \end{aligned} \tag{1}$$

In the usual dual formulation, firms choose prizes and an effort level so as to maximize a worker's expected utility subject to zero-profit and incentive compatibility conditions:

$$\max_{\substack{W_1 \geq W_2 \\ \mu \geq 0}} \quad \frac{U(W_1) + U(W_2)}{2} - C(\mu) \tag{2}$$

s.t.

$$\mu V = \frac{W_1 + W_2}{2} \tag{3}$$

$$\begin{aligned} & (U(W_1) - U(W_2))G(\widehat{\mu} - \mu) - C(\widehat{\mu}) \\ & \leq (U(W_1) - U(W_2))G(0) - C(\mu) \qquad (\widehat{\mu} \geq 0) \end{aligned} \tag{4}$$

Problem (2-4) will be called the *unrelaxed problem*. A solution (W_1^*, W_2^*, μ^*) to the unrelaxed problem will be referred to (somewhat loosely) as an *optimal tournament* associated with G.

Under the *first-order approach* (FOA), the continuum of incentive compatibility conditions in (4) is replaced by the single marginal condition

$$g(0)(U(W_1) - U(W_2)) = C'(\mu). \tag{5}$$

Condition (5) is necessary for any solution of the unrelaxed problem.[5] We will refer to the maximization problem (2), subject to constraints (3) and (5), as the *relaxed problem*.

[5]This is obvious if the optimum effort choice is interior, i.e., if $\mu^* > 0$. If, however, $\mu^* = 0$, then the Inada conditions imposed on the cost function imply that $W_1^* = W_2^*$, so that (5) is satisfied also in that case.

The relaxed problem is known to allow a solution $(W_1^{\text{FOA}}, W_2^{\text{FOA}}, \mu^{\text{FOA}})$ that can be approximated by replacing utility and cost functions with their respective second-order Taylor expansions. For example, the effort level and the prize spread may be approximated by

$$C'(\mu^{\text{FOA}}) \approx \frac{V}{1 + sC''/4g(0)^2} \tag{6}$$

and

$$W_1^{\text{FOA}} - W_2^{\text{FOA}} \approx \frac{g(0)V}{g(0)^2 + sC''/4}, \tag{7}$$

respectively, where $s = -U''/U'$ denotes the worker's Arrow-Pratt coefficient of absolute risk aversion, and marginal utility is normalized to unity at mean income.[6] Moreover, if the worker's expected utility function

$$U^{\text{FOA}}(\mu) = U(W_2^{\text{FOA}}) + G(\mu - \mu^{\text{FOA}})(U(W_1^{\text{FOA}}) - U(W_2^{\text{FOA}})) - C(\mu) \tag{8}$$

is, say, strictly concave, then $(W_1^{\text{FOA}}, W_2^{\text{FOA}}, \mu^{\text{FOA}})$ solves also the unrelaxed program. In particular, in the risk-neutral case, $C'(\mu^{\text{FOA}}) = V$, and the resulting allocation of resources is efficient.

When U^{FOA} is not strictly concave, however, then there is no guarantee that all the incentive compatibility conditions in (4) hold, i.e., the effort level μ^{FOA} may be merely a local maximum of U^{FOA}.[7] In other words, μ^{FOA} need not be a symmetric pure-strategy Nash equilibrium in the tournament with prizes $(W_1^{\text{FOA}}, W_2^{\text{FOA}})$. In that case, $(W_1^{\text{FOA}}, W_2^{\text{FOA}}, \mu^{\text{FOA}})$ will not be

[6] The specific expressions in (6) and (7) are taken from McLaughlin (1988, p. 231). These expressions are most accurate when $g(0)$ is large, so that the second-order Taylor approximations are accurate, and when s is small, so that the normalization of marginal utility matters least. When these conditions are not satisfied, however, it is preferable to solve the relaxed problem numerically, as done below.

[7] Indeed, μ^{FOA} may fail to be a global maximum of U^{FOA} even if the second-order condition holds strictly at μ^{FOA} (as it does under the present assumptions), and even if a deviation to a zero effort level is unprofitable for the worker.

a solution of the unrelaxed problem, which illustrates the limits of the first-order approach. In the present paper, a somewhat generous stance will be taken by calling the first-order approach *invalid* only if, for every solution of the relaxed problem, there is at least one incentive compatibility condition in (4) that fails to hold true.[8]

As pointed out by Green and Stokey (1983), the potential non-existence of a symmetric pure-strategy Nash equilibrium in a tournament with arbitrary prizes does not impair the firm's ability to design the contract (W_1, W_2, μ) in such a way that μ is a symmetric pure-strategy Nash equilibrium in the tournament with prizes (W_1, W_2).[9] In fact, as shown in the Appendix, this design problem can always be solved in an optimal way.

Proposition 1. *An optimal tournament exists (i.e., even if the first-order approach is invalid).*

The proposition raises the question of how the optimal tournament looks like in settings not traditionally considered. This question is addressed in the following sections.[10]

[8] Thus, the first-order approach is valid in the terminology of the present paper if *some* solution $(W_1^{\mathrm{FOA}}, W_2^{\mathrm{FOA}}, \mu^{\mathrm{FOA}})$ of the relaxed problem satisfies all the incentive compatibility conditions in (4). It will be noted that the definition for validity used in the present paper is slightly less demanding than the one employed by Rogerson (1985, p. 1363), who required for validity that *any* solution of the relaxed problem should satisfy incentive compatibility. However, the two definitions coincide when the relaxed problem has a unique solution, such as in the risk-neutral case or in the cases considered in Sections 4 and 5.

[9] See Green and Stokey (1983, fn. 3): "For arbitrary prize structures, there may be no Nash equilibrium, symmetric or otherwise. This is of no importance to us, since we are considering only tournaments that are *designed* so that they *do* have a symmetric Nash equilibrium (emphasis in the original)."

[10] The restriction to tournaments that allow a symmetric pure-strategy Nash equilibrium is definitely a choice we made. Alternatively, one could have assumed that firms may choose to implement mixed-strategy Nash equilibria. Unfortunately, the literature offers little guidance with regard to this point. For example, while Green and Stokey (1983)

3 Side-stepping the first-order approach

This section describes the envelope approach to rank-order tournaments that has been outlined in the Introduction. Note first that one may add the equality constraint

$$U(W_1) - U(W_2) = \Delta(\mu) \equiv \frac{C'(\mu)}{g(0)} \tag{9}$$

to the unrelaxed problem (2-4) without affecting the solution. Provided that (9) holds, however, incentive compatibility (4) becomes equivalent to

$$\Delta(\mu)(G(\widehat{\mu} - \mu) - G(0)) + C(\mu) - C(\widehat{\mu}) \leq 0 \qquad (\widehat{\mu} \geq 0). \tag{10}$$

Consider now the "upper envelope" of the individual constraints in (10), i.e.,

$$\varphi(\mu) = \max_{\widehat{\mu} \geq 0} \left\{ (G(\widehat{\mu} - \mu) - G(0))\Delta(\mu) + C(\mu) - C(\widehat{\mu}) \right\}, \tag{11}$$

where the maximum is attained as a consequence of the Inada conditions. The unrelaxed problem (2-4) may then be reformulated as

$$\max_{\mu \geq 0} \quad \overline{U}(\mu) \tag{12}$$

$$\text{s.t.} \quad \varphi(\mu) \leq 0, \tag{13}$$

where $\overline{U}(\mu)$ denotes the value of the firm's objective function (2) under the condition that the prize structure (W_1, W_2) is defined implicitly through (3) and (5).[11] The *reformulated problem* (12-13) is still not standard, because φ

consider only pure-strategy equilibria, Nalebuff and Stiglitz (1983) find it more natural to allow for randomization. This point will be taken up again in the conclusion.

[11] It is not hard to check that $\overline{U}(\mu)$ is well-defined for any $\mu \geq 0$. Indeed, using (3) to eliminate W_2 in (5), one obtains

$$U(W_1) - U(2V\mu - W_1) = \frac{C'(\mu)}{g(0)}.$$

Differentiating the left-hand side with respect to W_1, and noting that $U' > 0$, shows that there is at most one solution. Further, since $U'' \leq 0$, the left-hand side approaches $\pm\infty$ as $W_1 \to \pm\infty$. By continuity, there is a unique solution.

may have kinks. However, using the tools provided by Milgrom and Segal (2002), it can be shown that φ is monotone increasing if marginal costs are logconcave.[12] Moreover, since $\varphi(0) = 0$, monotonicity implies that the feasible set in the reformulated problem (12-13) is a closed interval whose left endpoint is zero. Hence, the following result is obtained.

Proposition 2. *Assume that C' is logconcave, and that the first-order approach is invalid. Then $\mu^* < \mu^{\mathrm{FOA}}$ for any pair of respective solutions of the unrelaxed and the relaxed problems.*

Proposition 2 shows that the first-order approach to tournament design is not innocuous, in the sense that it has the potential to cause a bias in the level of effort considered to be implementable.

For intuition, suppose that $(W_1^{\mathrm{FOA}}, W_2^{\mathrm{FOA}}, \mu^{\mathrm{FOA}})$ is a solution of the relaxed problem, yet that the first-order approach is invalid. Then, μ^{FOA} is not a symmetric pure-strategy Nash equilibrium in the tournament defined through prizes $(W_1^{\mathrm{FOA}}, W_2^{\mathrm{FOA}})$.[13] In other words, there necessarily exists an effort level $\mu_{\mathrm{FOA}} \neq \mu^{\mathrm{FOA}}$ such that $U^{\mathrm{FOA}}(\mu_{\mathrm{FOA}}) > U^{\mathrm{FOA}}(\mu^{\mathrm{FOA}})$, where U^{FOA} is defined through equation (8) above.[14] Without loss of generality, μ_{FOA} may be chosen to be a global optimum of U^{FOA}, so that $\varphi(\mu^{\mathrm{FOA}}) = U^{\mathrm{FOA}}(\mu_{\mathrm{FOA}}) - U^{\mathrm{FOA}}(\mu^{\mathrm{FOA}}) > 0$. But if φ is monotone increasing, then the

[12] Being a rather mild assumption, logconcavity of marginal costs has been imposed in prior work (e.g., Chan et al., 2009; Akerlof and Holden, 2012), and is consistent with both convex and concave marginal costs. Also, marginal costs cannot be globally logconvex under the Inada conditions imposed. Still, it remains an assumption, of course.

[13] Indeed, if μ^{FOA} were a symmetric pure-strategy Nash equilibrium in the tournament with prizes $(W_1^{\mathrm{FOA}}, W_2^{\mathrm{FOA}})$, then $(W_1^{\mathrm{FOA}}, W_2^{\mathrm{FOA}}, \mu^{\mathrm{FOA}})$ would satisfy incentive compatibility, and hence, would solve the unrelaxed problem, in conflict with our presumption that the first-order approach is invalid.

[14] Moreover, provided that g is unimodal, $\mu_{\mathrm{FOA}} < \mu^{\mathrm{FOA}}$, as intuition suggests.

firm's only way to reduce the worker's incentive to deviate is it to lower the contractual level of effort relative to μ^{FOA}.

To understand why an assumption on costs is needed, note that raising μ has altogether three effects on the envelope constraint (13). First, $C(\mu)$ increases, which tightens the constraint. Second, $U(W_1) - U(W_2)$ increases, which loosens the constraint. Finally, deviations become less likely to win, which also loosens (13). However, if costs are not excessively convex then the change to the prize structure remains sufficiently moderate compared to the differential of the other two effects, tipping the balance in favor of a tightening constraint.

The size of the potential welfare loss captured by Proposition 2 is not negligible. To the contrary, as will become clear below, tournaments may be quite ineffective as an incentive device.[15]

4 An explicit characterization

This section presents a complete characterization of the optimal tournament in a standard setting. Specifically, it will be assumed that costs are quadratic, i.e., that $C(\mu) = c\mu^2/2$ for some $c > 0$, and that workers exhibit a constant absolute risk aversion, i.e., that either $U(W) = -e^{-sW}/s$ for $s > 0$ or $U(W) = W$. These assumptions are made for tractability and can be relaxed. Indeed, as discussed below, the main features of the optimal tournament do not depend on these assumptions.

To describe the equilibrium in cases where the first-order approach is not

[15] To mitigate the welfare loss, firms might decide to use deliberately inaccurate performance measures (O'Keeffe et al., 1984), or to induce mixed-strategy equilibria (Nalebuff and Stiglitz, 1983, Appendix). Both options are excluded here, however.

valid, it proves useful to take a comparative statics perspective with respect to the dispersion of the differential error term. Thus, for a given distribution function G and an arbitrary parameter $\sigma > 0$, one defines a new distribution function $G_\sigma(z) \equiv G(z/\sigma)$, where a larger σ corresponds to a more dispersed distribution of the differential error term. E.g., if G is standard normal, then G_σ is normal with mean zero and standard deviation σ.

It is shown in the Appendix (see Lemma A.2) that, under the present assumptions, the firm's indirect objective function $\overline{U} \equiv \overline{U}_\sigma$ is strongly pseudo-concave in μ, i.e., that the strict second-order condition for a local maximum holds at any critical point. In particular, there is a unique optimal effort level $\mu^{\text{FOA}}(\sigma)$ in the relaxed problem associated with G_σ. As discussed in Section 2, this solution may be approximated in the case of risk aversion, and explicitly obtained in the case of risk neutrality. The optimal tournament $(W_1^*(\sigma), W_2^*(\sigma), \mu^*(\sigma))$ associated with G_σ may now be characterized in terms of $\mu^{\text{FOA}}(\sigma)$ as follows.

Proposition 3. *Suppose that costs are quadratic and that workers have CARA utility (which includes the case of risk-neutrality as a limit case). Then, there is a threshold value $\sigma^* > 0$ such that, for any $\sigma > 0$, the optimal tournament associated with G_σ is unique and implements the effort level*

$$\mu^*(\sigma) = \min\{\frac{\sigma}{\sigma^*} \cdot \mu^{\text{FOA}}(\sigma^*), \mu^{\text{FOA}}(\sigma)\}. \tag{14}$$

As the proposition shows, the optimal tournament will be shaped by the envelope constraint (13) once the level of individual-specific uncertainty falls below a certain level. In particular, the usual comparative statics result that

$\mu^{\text{FOA}}(\sigma)$ is monotone decreasing in σ,[16] is misleading about the comparative statics of μ^*. Instead, as illustrated in Figure 1, the optimally implemented effort level $\mu^* = \mu^*(\sigma)$ is strictly unimodal in the case of risk aversion. Similarly, μ^* is piecewise linear in the case of risk neutrality where μ^{FOA} is a constant.

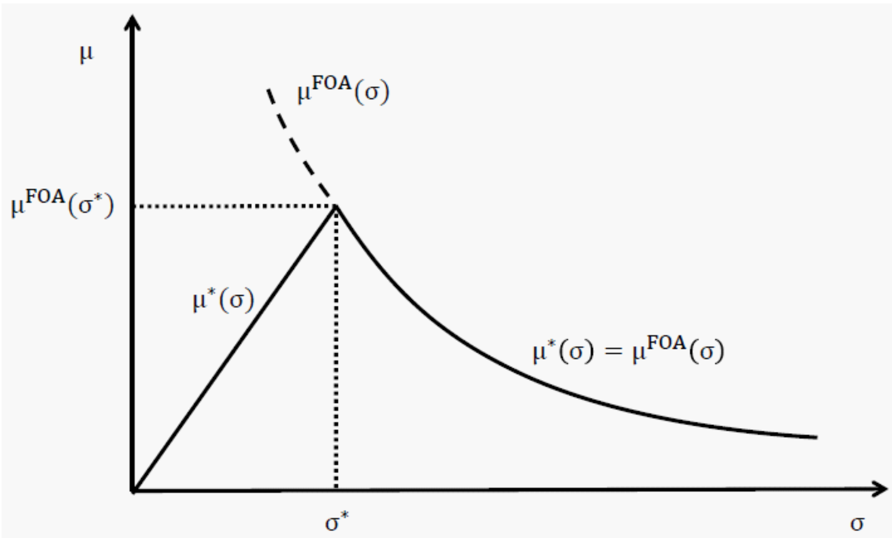

Figure 1. The optimally implemented effort level.

Denote by $W_1^{\text{FOA}}(\sigma)$ and $W_2^{\text{FOA}}(\sigma)$ the optimal prizes for the relaxed problem associated with G_σ. Using the second-order Taylor expansion of utility as above, the prize spread implementing the optimal effort level can be shown to satisfy

$$W_1^*(\sigma) - W_2^*(\sigma) \approx \min\{\frac{\sigma}{\sigma^*}, 1\} \cdot \left(W_1^{\text{FOA}}(\sigma) - W_2^{\text{FOA}}(\sigma)\right), \tag{15}$$

where the approximation is exact for $\sigma \geq \sigma^*$, and fairly precise for σ close

[16] Cf. Lazear and Rosen (1981, p. 853) and McLaughlin (1988, fn. 5). Under the specific assumptions of Proposition 3, the monotonicity of μ^{FOA} follows from Lemma A.3 in the Appendix.

to zero and s small.[17] Thus, also the predicted prize spread may be biased under the first-order approach. In particular, as σ gets smaller, the optimal spread diminishes much faster than the first-order approach would suggest.[18]

5 Discussion

To clarify what happens for $\sigma < \sigma^*$, consider a worker's expected utility from exerting an effort of μ in the optimal tournament associated with G_σ, i.e.,

$$U_\sigma^*(\mu) = U(W_2^*(\sigma)) + G_\sigma(\mu - \mu_*(\sigma))(U(W_1^*(\sigma)) - U(W_2^*(\sigma))) - C(\mu). \quad (16)$$

Then, the following observation can be made.

Remark 1. *For $\sigma < \sigma^*$, there is a "cheating level" $\mu_*(\sigma) \neq \mu^*(\sigma)$ such that $U_\sigma^*(\mu_*(\sigma)) = U_\sigma^*(\mu^*(\sigma))$.*

Thus, whenever the envelope constraint matters, the worker's objective function U_σ^* exhibits, besides its global maximum at $\mu^*(\sigma)$, at least one additional global maximum at some $\mu_*(\sigma) \neq \mu^*(\sigma)$. To see why this is so, suppose that there is no "cheating level." Then, as intuition suggests, the firm could marginally raise the contractual effort level above $\mu^*(\sigma)$, and still satisfy incentive compatibility.[19] But, by strong pseudoconcavity, the firm's indirect utility

[17] To see this, note that the necessary first-order condition (5) implies $W_1 - W_2 \approx c\mu\sigma/U'g(0)$ for the respective solutions of the unrelaxed and the relaxed problems.

[18] When the assumptions of Proposition 3 are relaxed, one can still show that $\mu^*(\sigma) = \mu^{\mathrm{FOA}}(\sigma)$ for σ sufficiently large and that $\mu^*(\sigma) \to 0$ as $\sigma \to 0$. Thus, even though the homogeneous relationships reflected in (14) and (15) tend to break down for cost functions that do not exhibit a constant elasticity, the characterization result captures, in its essence, a more general fact.

[19] Given that the worker's local second-order condition holds strictly at $\mu^*(\sigma)$, this point turns out to be an immediate consequence of Berge's Theorem.

function is strictly increasing at $\mu^*(\sigma)$, leading to a contradiction. Hence, the worker's best-response set indeed consists of at least two elements.[20]

For a general density g, there may be many "cheating levels," possibly infinitely many. For g sufficiently well-behaved, however, it turns out that there is at most one global maximizer of U_σ^* other than $\mu^*(\sigma)$. We will say that g is *strictly bell-shaped* if there is an $r > 0$ such that $g''(z) \gtrless 0$ if $|z| \gtrless r$. The following observation can now be made.

Remark 2. *Suppose that g is strictly bell-shaped. Then, for any $\sigma \leq \sigma^*(s)$,*

$$\mu^*(\sigma) = \frac{\sigma\gamma g(0)}{g(0) - g(\gamma)}, \tag{17}$$

$$\mu_*(\sigma) = \frac{\sigma\gamma g(\gamma)}{g(0) - g(\gamma)}, \tag{18}$$

where γ is the unique strictly positive solution of the equation

$$\frac{g(0) + g(\gamma)}{2} = \frac{1}{\gamma}\int_0^\gamma g(z)dz. \tag{19}$$

The two remarks above are illustrated by the following two examples.

Example 1. For g standard normal, $\gamma = 2.2809$. Hence, $\mu_*(\sigma) = 0.1827 \cdot \sigma$ and $\mu^*(\sigma) = 2.4636 \cdot \sigma$, for any $\sigma \leq \sigma^*$. For $s = 0$, this implies $\mu^*(\sigma) = \min\{2.4637 \cdot \sigma; V/c\}$, so that $\sigma^* = 0.4059 \cdot V/c$. For $s = 0.5, 1, 2, 3, 10$, the relaxed problem was solved numerically over the grid $\sigma = 0.01, ..., 1.00$. On that sample, $\sigma^*(s)$ was found to be strictly declining in s, which is intuitive.

[20]The necessity of a "cheating level" may be familiar from Grossman and Hart (1983, Prop. 6) or Mookherjee (1984, Prop. 1). There, the absence of a utility-equivalent lower level of effort would allow the principal to implement the same level of effort at lower cost. Here, similarly, even though actions are continuous, the absence of a "cheating level" would allow the firm to implement a higher level of effort.

Example 2. If ε_1 and ε_2 are uniformly distributed, then ξ follows a triangular distribution (e.g., Bull et al., 1987; Altmann et al., 2012). The normalized density $g(z) = \max\{0, \min\{1+z, 1-z\}\}$ is, however, not strictly bell-shaped, so that Remark 2 does not apply. Still, the conclusion of Proposition 3 holds. For instance, for $s = 0$, one can check that $\varphi(\mu) \equiv \varphi_\sigma(\mu) = \frac{c\mu}{2} \max\{0, \mu - \sigma\}$, so that the optimal tournament associated with G_σ is characterized by $\mu^*(\sigma) = \min\{\sigma, V/c\}$ and $W^*_{1/2}(\sigma) = (V \pm \frac{c\sigma}{2})\mu^*(\sigma)$.

Notably, the envelope constraint (13) may come into play in response to changes in V, c, or s, i.e., even if the information structure does not change. As discussed in the next section, an increase in the number of contestants may have a similar effect.

6 Large tournaments

This section considers an extension to tournaments with more than two contestants. Attention will be restricted to the case of a single winner.

Denote by F and f the distribution and density functions associated with an individual error term ε (assumed i.i.d. across players). Considering a tournament between n workers, and provided that all opponents of some given player j exert the same effort level μ, worker j's probability of winning may be represented as

$$G_n(\mu_j, \mu) = \int_{-\infty}^{+\infty} F(\mu_j + \varepsilon - \mu)^{n-1} dF(\varepsilon). \tag{20}$$

The problem of the firm is only slightly modified:

$$\max_{\substack{W_1 \geq W_2 \\ \mu \geq 0}} \quad \frac{U(W_1) + (n-1)U(W_2)}{n} - C(\mu) \tag{21}$$

s.t.

$$\mu V = \frac{W_1 + (n-1)W_2}{n} \tag{22}$$

$$(U(W_1) - U(W_2))G_n(\widehat{\mu}, \mu) - C(\widehat{\mu}) \tag{23}$$

$$\leq (U(W_1) - U(W_2))G_n(\widehat{\mu}, \mu) - C(\mu) \qquad (\widehat{\mu} \geq 0)$$

The optimal tournament satisfies, in particular, the necessary first-order condition for a symmetric pure-strategy Nash equilibrium,

$$U(W_1) - U(W_2) = \Delta_n(\mu) \equiv \frac{C'(\mu)}{g_n}, \tag{24}$$

where

$$g_n = (n-1) \int_{-\infty}^{+\infty} F(\varepsilon)^{n-2} f(\varepsilon)^2 d\varepsilon. \tag{25}$$

An approximation for the solution of the relaxed problem, μ_n^{FOA}, can be found as before. However, as pointed out by McLaughlin (1988, p. 241), it is in general very difficult to tell if the first-order approach is valid for large n.

To side-step the first-order approach, one defines again the "upper envelope," which reads in this case

$$\varphi_n(\mu) = \max_{\widehat{\mu} \geq 0} \left\{ (G_n(\widehat{\mu}, \mu) - G_n(\mu, \mu))\Delta_n(\mu) + C(\mu) - C(\widehat{\mu}) \right\}. \tag{26}$$

Then, as above, one can show that if marginal costs are logconcave, then the optimally implemented effort μ_n^* in the tournament between n workers and the corresponding optimal effort level μ_n^{FOA} in the relaxed problem satisfy $\mu_n^* \leq \mu_n^{\mathrm{FOA}}$. Thus, also in tournaments with more than two contestants, the first-order approach, if invalid, would tend to overstate implemented effort levels.

Additional conclusions can be obtained by focusing, as Nalebuff and Stiglitz (1983) do, on the incentive compatibility condition at the specific effort level $\widehat{\mu} = 0$. In the case of the normal distribution at least, one may then characterize the limit behavior of μ_n^* as follows.

Proposition 4. *Suppose that F is normal. Then, as the number of contestants n increases above all finite bounds, the optimally implemented effort level μ_n^* goes to zero.*

The result above characterizes the limit behavior of a sequence of optimal tournaments in a setting where it is a priori not clear if the first-order approach is applicable. It follows from the proposition that the first-order approach is indeed invalid in large tournaments in the case of risk-neutrality. Even though Proposition 4 holds also under the assumption of risk-aversion, no conclusion is possible about the validity of the first-order approach in large tournaments for the case of risk-aversion. However, this fact only supports the usefulness of the envelope approach because it delivers results also in situations where sufficient conditions for the first-order approach may be difficult to find.

7 Conclusion

It has been shown that the first-order approach, if used exclusively, may lead to a positively biased assessment of the efficiency of rank-order tournaments. In particular, tournaments may not be very suitable as compensation schemes when performance is a relatively good signal of effort. Intuitively, prize spread and performance measurement are complements, forcing firms to reduce the former when the latter improves. In the settings studied above, the prize structure is so unrewarding that the avoidance of cheating becomes a binding constraint, overruling the usual trade-off between risk and incentives. As a consequence, individual contracts such as piece rates may dominate the optimal tournament even when workers are risk-neutral.[21]

In a recent survey, Waldman (2013) finds as one of the testable predictions of tournament theory that the prize is increasing in the number of contestants. The results of the present paper suggest, however, that that prediction might not be robust because with many contestants, the first-order approach need not be valid, and the symmetric pure-strategy equilibrium may lead to inefficient levels of effort. This observation might even help to explain the *negative* relationship between the salary gap between CEO and vice president and the number of VPs (O'Reilly et al., 1988).

Regarding further research, one issue might be the question of whether the theoretical issues discussed in this paper may constitute a practical reason for not using tournaments. For example, Lazear and Rosen (1981, p. 848) argue that in the case of risk-neutrality, the tie between individual contracts and tournaments is broken by differential costs of information and measurement. The present analysis obviously provides an alternative hypothesis. Another interesting issue would be the extension of the present analysis to tournaments with more than a single winner (Krishna and Morgan, 1998; Kalra

[21] With this type of observation, the present paper takes the same line as, e.g., Chaigneau et al. (2014), who show that the sufficient statistics theorem fails to hold when the first-order approach is dropped in a standard principal-agent problem.

and Shi, 2001; Budde, 2009; Akerlof and Holden, 2012) or to various types of unbalanced tournaments (e.g., Meyer, 1992; Kono and Yagi, 2008; Imhof and Kräkel, 2015). Finally, it might be worthwhile to explore whether the comparably simple approach outlined in Section 3 could be applied to other settings in contract theory and mechanism design.[22]

Appendix

Proof of Proposition 1. By Jensen's inequality, constraint (3) implies $(U(W_1) + U(W_2))/2 \leq U(\mu V)$. Hence, from the Inada conditions, there is a $\overline{\mu} > 0$ such that implementing any $\mu > \overline{\mu}$ is strictly inferior to $\mu = 0$. By (3) and (5), this implies that there is a $\overline{W} > 0$ such that $W_1, W_2 \in [-\overline{W}, \overline{W}]$ for any optimal tournament. Thus, one may replace the feasible set by $I = \{(W_1, W_2, \mu) \in [-\overline{W}, \overline{W}]^2 \times [0, \overline{\mu}] : (3), (4), \text{ and } W_1 \geq W_2\}$. But $I \neq \varnothing$, because $(0, 0, 0) \in I$. Moreover, I is closed as an intersection of closed sets. $\square$

The following lemma is used in the proof of Proposition 2.

[22]The present paper has followed Green and Stokey (1983) in assuming that firms restrict attention to tournaments that have a symmetric pure-strategy Nash equilibrium. Under this assumption, tournaments ultimately become useless as an incentive device when monitoring becomes arbitrarily precise. Indeed, in the limit, there does not exist any pure-strategy equilibrium for $W_1 > W_2$ (e.g., Bull et al., 1987, fn. 3), forcing the firm to set $W_1 = W_2$. However, allowing for randomization would not re-establish efficiency. Also, the characterization of mixed-strategy equilibria in tournaments with little noise seems to require different methods (Ewerhart, 2015), and therefore lies beyond the scope of the present analysis.

Lemma A.1. *Define*

$$\psi(\mu,\widehat{\mu}) \equiv \frac{\partial}{\partial\mu}\left\{\Delta(\mu)(G(\widehat{\mu}-\mu)-G(0))+C(\mu)-C(\widehat{\mu})\right\} \tag{27}$$

$$= \Delta'(\mu)(G(\widehat{\mu}-\mu)-G(0))-\Delta(\mu)g(\widehat{\mu}-\mu)+C'(\mu), \tag{28}$$

where $\Delta'(\mu) = C''(\mu)/g(0)$*. Then the family* $\{\psi(\cdot,\widehat{\mu})\}_{\widehat{\mu}\geq 0}$ *is equidifferentiable at any* $\mu \geq 0$.

Proof. Since g is a density with bounded first and second derivatives,

$$\frac{\partial^2\psi(\mu,\widehat{\mu})}{\partial\mu^2} = \Delta'''(\mu)(G(\widehat{\mu}-\mu)-G(0))-3\Delta''(\mu)g(\widehat{\mu}-\mu) \tag{29}$$

$$+3\Delta'(\mu)g'(\widehat{\mu}-\mu)-\Delta(\mu)g''(\widehat{\mu}-\mu)+C'''(\mu) \tag{30}$$

exists and is bounded in $\widehat{\mu}$, for any $\mu \geq 0$. It follows that the family $\{\partial\psi(\cdot,\widehat{\mu})/\partial\mu\}_{\widehat{\mu}\geq 0}$ is equicontinuous at any $\mu \geq 0$. Using the Mean Value Theorem, as in Milgrom and Segal (2002, p. 587), $\{\psi(\cdot,\widehat{\mu})\}_{\widehat{\mu}\geq 0}$ is now seen to be equidifferentiable at any $\mu \geq 0$. □

Proof of Proposition 2. Denote by $X(\mu)$ the set of maximizers in problem (11). Using Lemma A.1, it follows from Milgrom and Segal (2002, Th. 1&3) that φ is right-hand differentiable at any $\mu \geq 0$ with

$$\varphi'(\mu+) \equiv \lim_{\varepsilon\to 0+}\frac{1}{\varepsilon}\left(\varphi(\mu+\varepsilon)-\varphi(\mu)\right) \geq \psi(\mu,\widehat{\mu}), \tag{31}$$

for any $\widehat{\mu} \in X(\mu)$.[23] Moreover, as a consequence of local and global optimality conditions,

$$\Delta(\mu)g(\widehat{\mu}-\mu)-C'(\widehat{\mu}) \leq 0, \tag{32}$$

[23] Intuitively, the value function increases by at least as much as the value at any given global maximum.

and

$$\Delta(\mu)(G(\widehat{\mu}-\mu)-G(0))+C(\mu)-C(\widehat{\mu})\geq 0, \tag{33}$$

for any $\widehat{\mu}\in X(\mu)$. Suppose $\mu>0$. Then, using inequalities (32) and (33) to put a lower bound on (28) shows that

$$\varphi'(\mu+)\geq -\frac{C''(\mu)}{C'(\mu)}(C(\mu)-C(\widehat{\mu}))-C'(\widehat{\mu})+C'(\mu)\equiv\phi(\mu,\widehat{\mu}) \tag{34}$$

for any $\widehat{\mu}\in X(\mu)$. By assumption, C''/C' is weakly decreasing. Therefore, for any $\widehat{\mu}\leq\mu$,

$$\begin{aligned}\frac{C''(\mu)}{C'(\mu)}(C(\mu)-C(\widehat{\mu})) &= \frac{C''(\mu)}{C'(\mu)}\int_{\widehat{\mu}}^{\mu}C'(\widetilde{\mu})d\widetilde{\mu} &(35)\\ &\leq \int_{\widehat{\mu}}^{\mu}C'(\widetilde{\mu})\frac{C''(\widetilde{\mu})}{C'(\widetilde{\mu})}d\widetilde{\mu} &(36)\\ &= C'(\mu)-C'(\widehat{\mu}). &(37)\end{aligned}$$

Hence, $\phi(\mu,\widehat{\mu})\geq 0$ in this case. Using completely analogous arguments, one shows that, similarly, $\phi(\mu,\widehat{\mu})\geq 0$ if $\widehat{\mu}>\mu$. Thus, $\varphi'(\mu+)\geq 0$ for any $\mu>0$. Note also that φ is continuous on $\mathbb{R}_+$, as a consequence of Berge's theorem. It follows that φ is monotone increasing (Royden, 1988, Sec. 5). Hence, noting that $\varphi(0)=0$, the feasible set of problem (12-13) is an interval $[0,\mu^{\#}]$, for some $\mu^{\#}\geq 0$. But μ^{FOA} is a global optimum of $\overline{U}$. Therefore, $\mu^*\leq\mu^{\mathrm{FOA}}$, proving the first assertion. The second assertion is now immediate. □

For the following three lemmas, the assumptions of Proposition 3 are imposed.

Lemma A.2. *$\overline{U}$ is strongly pseudoconcave in μ.*

Proof. Total differentiation of equations (3) and (5), and subsequently solving the resulting system of linear equations, yields

$$\frac{dW_1}{d\mu} = \frac{2Vu_2' + c/g(0)}{u_1' + u_2'}, \tag{38}$$

$$\frac{dW_2}{d\mu} = \frac{2Vu_1' - c/g(0)}{u_1' + u_2'}, \tag{39}$$

where $u_1' \equiv U'(W_1)$ and $u_2' \equiv U'(W_2)$. Therefore,

$$\frac{\partial \overline{U}}{\partial \mu} = 2V\frac{u_1'u_2'}{u_1' + u_2'} + \frac{c}{2g(0)}\frac{u_1' - u_2'}{u_1' + u_2'} - c\mu. \tag{40}$$

Differentiating (40) with respect to μ, and assuming that $\partial\overline{U}/\partial\mu = 0$, one obtains

$$\begin{aligned}\frac{\partial^2 \overline{U}}{\partial \mu^2} &= \frac{2V}{u_1' + u_2'}\left\{u_1''u_2'\frac{dW_1}{d\mu} + u_1'u_2''\frac{dW_2}{d\mu}\right\} + \frac{c/2g(0)}{u_1' + u_2'}\left\{u_1''\frac{dW_1}{d\mu} - u_2''\frac{dW_2}{d\mu}\right\} \\ &\quad - \frac{c\mu}{u_1' + u_2'}\cdot\left\{u_1''\frac{dW_1}{d\mu} + u_2''\frac{dW_2}{d\mu}\right\} - c, \end{aligned} \tag{41}$$

where $u_1'' \equiv U''(W_1)$ and $u_2'' \equiv U''(W_2)$. Hence, using (38-39) and $\partial\overline{U}/\partial\mu = 0$ another time, one arrives at

$$\frac{\partial^2 \overline{U}}{\partial \mu^2} = (-2s)\cdot\frac{2V^2u_1'u_2' + c^2/4g(0)^2 - c^2\mu^2}{u_1' + u_2'} - c, \tag{42}$$

where $s = -u_1''/u_1' = -u_2''/u_2' \geq 0$. It follows that $\partial^2\overline{U}/\partial\mu^2 < 0$ if $\mu \leq 1/2g(0)$. Otherwise, i.e., if $\mu > 1/2g(0)$, then $\partial\overline{U}/\partial\mu = 0$ implies

$$2Vu_1'u_2' = c\mu(u_1' + u_2') - \frac{c}{2g(0)}(u_1' - u_2') \tag{43}$$

$$= c(\mu - \frac{1}{2g(0)})u_1' + c(\mu + \frac{1}{2g(0)})u_2' \tag{44}$$

$$\geq c\mu u_2'. \tag{45}$$

Hence,

$$2Vu_1' \geq c\mu. \tag{46}$$

Moreover, combining (43) with $u_2' \geq u_1'$, one finds $2Vu_1'u_2' \geq 2c\mu u_1'$, so that

$$Vu_2' \geq c\mu. \tag{47}$$

Multiplying the two inequalities (46) and (47), one arrives at $2V^2u_1'u_2' \geq c^2\mu^2$. It follows that $\partial^2\overline{U}/\partial\mu^2 < 0$, which proves the claim. $\square$

Lemma A.3. $\mu^{\text{FOA}} > 0$*. Moreover,* μ^{FOA} *is continuous and, provided* $s > 0$*, strictly decreasing in* σ*.*

Proof. It is shown first that $\mu^{\text{FOA}} > 0$. Indeed, for $\mu = 0$, equations (3) and (5) imply $W_1 = W_2$, so that evaluating equation (40) at $\mu = 0$ yields

$$\left.\frac{\partial\overline{U}}{\partial\mu}\right|_{\mu=0} = 2V\frac{u_1'u_2'}{u_1'+u_2'} > 0. \tag{48}$$

Hence, $\mu^{\text{FOA}} > 0$, as claimed. Differentiating now (40) with respect to σ and exploiting that $\partial\overline{U}/\partial\mu = 0$, one obtains

$$\begin{aligned}\frac{\partial^2\overline{U}}{\partial\mu\partial\sigma} = {} & \frac{2V}{u_1'+u_2'}\left\{u_1''u_2'\frac{dW_1}{d\sigma} + u_1'u_2''\frac{dW_2}{d\sigma}\right\} + \frac{c\sigma/2g(0)}{u_1'+u_2'}\left\{u_1''\frac{dW_1}{d\sigma} - u_2''\frac{dW_2}{d\sigma}\right\} \\ & - \frac{c\mu}{u_1'+u_2'}\cdot\left\{u_1''\frac{dW_1}{d\sigma} + u_2''\frac{dW_2}{d\sigma}\right\} + \frac{c}{2g(0)}\frac{u_1'-u_2'}{u_1'+u_2'}.\end{aligned} \tag{49}$$

But, from equation (3) and the first-order condition $U(W_1) - U(W_2) = \frac{c\mu\sigma}{g(0)}$, it is immediate that

$$\frac{dW_1}{d\sigma} = -\frac{dW_2}{d\sigma} = \frac{c\mu}{g(0)(u_1'+u_2')}. \tag{50}$$

Simplifying the right-hand side of (49) using (50), and using that $s > 0$, one arrives at

$$\frac{\partial^2\overline{U}}{\partial\mu\partial\sigma} = -\frac{s\sigma c^2\mu}{2g(0)^2(u_1'+u_2')} - \frac{sc^2\mu^2(u_2'-u_1')}{g(0)(u_1'+u_2')^2} - \frac{c(u_2'-u_1')}{2g(0)(u_1'+u_2')} < 0. \tag{51}$$

Since $\overline{U}$ is strongly pseudoconcave with respect to μ, the claim follows. $\square$

Lemma A.4. $\mu^*(\sigma) \neq \mu^{\text{FOA}}(\sigma)$ *for some* $\sigma > 0$.

Proof. From incentive compatibility with respect to a deviation to $\widehat{\mu} = 0$,

$$0 \geq (G_\sigma(-\mu) - G_\sigma(0))\frac{C'(\sigma)}{g_\sigma(0)} + C(\mu) - C(0) \geq -\frac{c\sigma\mu}{g(0)} + \frac{c\mu^2}{2}, \tag{52}$$

where the second inequality follows from $G_\sigma \leq 1$. Hence, $\mu \leq 2\sigma/g(0)$, and therefore, $\mu^*(\sigma) \to 0$ as $\sigma \to 0$. On the other hand, by Lemma A.3, $\mu^{\text{FOA}}(\sigma)$ does not tend to zero as $\sigma \to 0$. Thus, for $\sigma > 0$ sufficiently small, $\mu^*(\sigma) \neq \mu^{\text{FOA}}(\sigma)$, which proves the lemma. $\square$

Proof of Proposition 3. By Lemma A.4, there is a $\widetilde{\sigma} > 0$ such that $\mu^*(\widetilde{\sigma}) \neq \mu^{\text{FOA}}(\widetilde{\sigma})$. Hence, the envelope constraint must be binding in the reformulated problem associated with $G_{\widetilde{\sigma}}$. Since marginal costs are logconcave, it follows from the proof of Proposition 2 that $\mu \leq \mu^*(\widetilde{\sigma})$ is equivalent to

$$(G_{\widetilde{\sigma}}(\widehat{\mu} - \mu) - G_{\widetilde{\sigma}}(0))\frac{C'(\mu)}{g_{\widetilde{\sigma}}(0)} + C(\mu) - C(\widehat{\mu}) \leq 0 \quad (\widehat{\mu} \geq 0). \tag{53}$$

Let $\sigma > 0$. Then, with $\lambda \equiv \sigma/\widetilde{\sigma}$, purely algebraic manipulation exploiting the homogeneity of the cost function shows that

$$\begin{aligned} &(G_{\widetilde{\sigma}}(\widehat{\mu} - \mu) - G_{\widetilde{\sigma}}(0))\frac{C'(\mu)}{g_{\widetilde{\sigma}}(0)} + C(\mu) - C(\widehat{\mu}) \\ &= \frac{1}{\lambda^2}\left\{(G_\sigma(\widehat{\mu}_\lambda - \mu_\lambda) - G_\sigma(0))\frac{C'(\mu_\lambda)}{g_\sigma(0)} + C(\mu_\lambda) - C(\widehat{\mu}_\lambda)\right\}, \end{aligned} \tag{54}$$

where $\mu_\lambda \equiv \lambda\mu$ and $\widehat{\mu}_\lambda \equiv \lambda\widehat{\mu}$. Hence, $\mu_\lambda \leq \lambda\mu^*(\widetilde{\sigma})$ is equivalent to

$$(G_\sigma(\widehat{\mu}_\lambda - \mu_\lambda) - G_\sigma(0))\frac{C'(\mu_\lambda)}{g_\sigma(0)} + C(\mu_\lambda) - C(\widehat{\mu}_\lambda) \leq 0 \quad (\widehat{\mu}_\lambda \geq 0). \tag{55}$$

Invoking Lemma A.2, it follows that

$$\mu^*(\sigma) = \min\{\frac{\sigma}{\widetilde{\sigma}}\mu^*(\widetilde{\sigma}), \mu^{\text{FOA}}(\sigma)\} \tag{56}$$

for any $\sigma > 0$. By Lemma A.3, there is a unique σ^* such that

$$\frac{\sigma^*}{\widetilde{\sigma}}\mu^*(\widetilde{\sigma}) = \mu^{\text{FOA}}(\sigma^*). \tag{57}$$

Moreover,

$$\mu^*(\sigma) = \frac{\sigma}{\widetilde{\sigma}}\mu^*(\widetilde{\sigma}) = \frac{\sigma}{\sigma^*}\mu^{\text{FOA}}(\sigma^*) \tag{58}$$

if $\sigma \leq \sigma^*$, and $\mu^*(\sigma) = \mu^{\text{FOA}}(\sigma)$ if $\sigma > \sigma^*$. $\square$

Proof of Remark 1. Suppose the firm intends to implement an effort level $\widetilde{\mu}$. In the resulting tournament, a worker's expected utility from exerting an effort of μ may be written as

$$\widetilde{U}_\sigma(\mu|\widetilde{\mu}) = \underline{U}_\sigma(\widetilde{\mu}) + G_\sigma(\mu - \widetilde{\mu})\Delta_\sigma(\widetilde{\mu}) - C(\mu), \tag{59}$$

where $\Delta_\sigma(\widetilde{\mu}) = c\widetilde{\mu}/g_\sigma(0)$, and $\underline{U}_\sigma$ is a function that does not depend on μ. Note that

$$\frac{\partial^2 \widetilde{U}_\sigma(\mu|\widetilde{\mu})}{\partial \mu^2} = \frac{g'_\sigma(\mu - \widetilde{\mu})c\widetilde{\mu}}{g_\sigma(0)} - c. \tag{60}$$

Since g'_σ is continuous with $g'_\sigma(0) = 0$, this implies that there is an open and bounded neighborhood $\mathcal{N}$ of $\mu^*(\sigma)$ such that (60) is strictly negative for any $(\mu, \widetilde{\mu}) \in \mathcal{N} \times \mathcal{N}$. In particular, for any $\widetilde{\mu} \in \mathcal{N}$, the restriction of $\widetilde{U}_\sigma(\cdot|\widetilde{\mu})$ to $\mathcal{N}$ has a unique maximum at $\widetilde{\mu}$. Since $\mathcal{N}$ is bounded, Inada conditions imply there is some $\mu^{\max} > 0$ such that $\widetilde{U}_\sigma(\mu|\widetilde{\mu}) < \widetilde{U}_\sigma(\widetilde{\mu}|\widetilde{\mu})$ for any $\mu > \mu^{\max}$ and for any $\widetilde{\mu} \in \mathcal{N}$. By choosing the open set $\mathcal{N}$ sufficiently small, the compact set $\mathcal{M} = [0, \mu^{\max}]\backslash\mathcal{N}$ is clearly non-empty. The restriction of $U_\sigma^* \equiv \widetilde{U}_\sigma(\cdot|\mu^*(\sigma))$ to $\mathcal{M}$ therefore assumes its maximum in $\mathcal{M}$, say at some $\mu_*(\sigma)$. Incentive compatibility implies $U_\sigma^*(\mu_*(\sigma)) - U_\sigma^*(\mu^*(\sigma)) \leq 0$. To provoke a contradiction, suppose that $U_\sigma^*(\mu_*(\sigma)) - U_\sigma^*(\mu^*(\sigma)) < 0$. Then, $\widetilde{U}_\sigma(\mu|\mu^*(\sigma)) - \widetilde{U}_\sigma(\mu^*(\sigma)|\mu^*(\sigma)) < 0$ for any $\mu \in \mathcal{M}$. By Berge's Theorem, this implies $\widetilde{U}_\sigma(\mu|\widetilde{\mu}) - \widetilde{U}_\sigma(\mu^*(\sigma)|\widetilde{\mu}) < 0$ for any $\mu \in \mathcal{M}$, provided that $\widetilde{\mu}$ is sufficiently close to $\mu^*(\sigma)$. Hence, such $\widetilde{\mu}$ is incentive compatible. But, by Lemma A.2, $\overline{U}_\sigma(\widetilde{\mu})$ is strictly increasing in $\widetilde{\mu}$ in a neighborhood of $\mu^*(\sigma)$, which is impossible. □

Proof of Remark 2. W.l.o.g., let $\sigma < \sigma^*$ (the case $\sigma = \sigma^*$ follows by continuity), and choose some optimal $(W_1^*(\sigma), W_2^*(\sigma), \mu^*(\sigma))$ associated with G_σ. Then, from Remark 1, there is a $\mu_*(\sigma)$ such that

$$(U(W_1^*) - U(W_2^*))(G_\sigma(\mu_* - \mu^*) - G_\sigma(0)) = \frac{c}{2}\left\{(\mu_*)^2 - (\mu^*)^2\right\}, \qquad (61)$$

where the argument σ has been dropped to ease notation. From the Inada conditions, $\mu_* > 0$. Hence, both μ^* and μ_* are interior maxima of $U_\sigma^*(\cdot|, \mu^*(\sigma))$, satisfying

$$g_\sigma(0)(U(W_1^*) - U(W_2^*)) = c\mu^*, \qquad (62)$$

$$g_\sigma(\mu_* - \mu^*)(U(W_1^*) - U(W_2^*)) = c\mu_*, \qquad (63)$$

where $g_\sigma(z) = g(z/\sigma)/\sigma$. Adding (62) and (63) up, multiplying the result through with $(\mu^* - \mu_*)/2$, and subsequently subtracting (61), one arrives at

$$\frac{g_\sigma(0) + g_\sigma(\gamma)}{2} = \frac{G_\sigma(\gamma) - G_\sigma(0)}{\gamma}, \qquad (64)$$

where $\gamma = \mu^* - \mu_*$. Equation (64) allows at most one strictly positive solution $\gamma = \gamma(\sigma)$. To see this, define the function

$$h_\sigma(\gamma) = G_\sigma(\gamma) - G_\sigma(0) - \frac{g_\sigma(0) + g_\sigma(\gamma)}{2}\gamma. \qquad (65)$$

Then, $h'_\sigma(0) = 0$, and $h''(\gamma) = -\gamma g''(\gamma)/2$. Thus, since g is strictly bell-shaped, there is indeed at most one solution. Next, note that (64) implies $\gamma(\sigma) = \sigma \cdot \gamma(1)$. Finally, from the first-order conditions, $g(0)/g(\gamma) = \mu^*(\sigma)/\mu_*(\sigma)$. Simple algebra leads now to (17). □

Proof of Proposition 4. Consider the specific deviation to $\widehat{\mu} = 0$. For any $\mu \geq 0$, we have

$$\varphi_n(\mu) \geq (G_n(0,\mu) - G_n(\mu,\mu))\frac{C'(\mu)}{g_n} + C(\mu) - C(0) \tag{66}$$

$$\geq -\frac{C'(\mu)}{ng_n} + C(\mu) - C(0), \tag{67}$$

since $G_n(\mu,\mu) = \frac{1}{n}$. For μ_n^* to constitute an equilibrium in the tournament between n workers, it is necessary that $\varphi_n(\mu_n^*) \leq 0$. Hence,

$$\frac{C(\mu_n^*) - C(0)}{C'(\mu_n^*)} \leq \frac{1}{ng_n}. \tag{68}$$

Because $f'(\varepsilon) = -\varepsilon f(\varepsilon)/\sigma^2$ in the case of the normal distribution, integrating by parts yields

$$ng_n = n\int_{-\infty}^{+\infty} (n-1)F(\varepsilon)^{n-2}f(\varepsilon)^2 d\varepsilon \tag{69}$$

$$= -n\int_{-\infty}^{+\infty} F(\varepsilon)^{n-1}f'(\varepsilon)d\varepsilon \tag{70}$$

$$= \frac{n}{\sigma^2}\int_{-\infty}^{+\infty} \varepsilon F(\varepsilon)^{n-1}f(\varepsilon)d\varepsilon \tag{71}$$

$$\backsimeq \frac{1}{\sigma}\sqrt{2\ln n}, \tag{72}$$

where the asymptotic relationship for the mean extreme of n identically and independently distributed normal variables has been taken from David and Nagaraja (2003, Sec. 10.5). But, as in the proof of Proposition 1, Jensen's inequality implies

$$\frac{U(W_1) + (n-1)U(W_2)}{n} \leq U(\mu V) \tag{73}$$

for any n. Hence, $\mu_n^* \leq \overline{\mu}$ for any n. Since $ng_n \to \infty$ for $n \to \infty$, it follows from (68) that, indeed, $\mu_n^* \to 0$ for $n \to \infty$. $\square$

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